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The problem of the thermal conductivity equation in the presence of radiative energy transfer

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In the presence of the radiant thermal conductivity in a solid body the equation of the heat transfer and the boundary condition for the case of the surface radiation are shown to involve the different coefficients of the radiant thermal conductivity. The radiant thermal conductivity, in boundary conditions, is $2n_i^2/(1-\bar{R}_i)\,f(n)$ times less the analogous coefficient in a differential equation, n and R_i being the refraction and reflexion coefficients of the infrared waves in a body, respectively.

In quite a number of works (see, for example, Franck-Kamenetsky 1959; Filippov 1955) under definite conditions, the energy transfer in a body by radiation is shown to be characterized by the radiant thermal conductivity coefficient K_r, being analogous to the ordinary molecular thermal conductivity K_0.

In this case the heat transfer in the body may be described by the equation

$$\frac{\partial}{\partial x}\left[(K_0+K_r)\frac{\partial T}{\partial x}\right]=\rho c\frac{\partial T}{\partial t}, \tag{1}$$

where ρ is the density, c, the heat capacity of the material, T, the temperature. Under the ordinary temperature conditions for the solid bodies $K_0 \gg K_r$. The inverse ratio takes place for the conditions within the stars. Equation (1) is only valid when the length of the free quantum path of a medium heat radiation is small in comparison with all the characteristic lengths of a problem (the length of a temperature wave, the size of inhomogeneities, etc.). These conditions are usually realized in a solid body.

The given equation with the boundary conditions of the radiant energy balance on a surface is used, very often, for the calculation of the thermal régime of the Moon and planets, without the atmosphere (Ingrao, Young & Linsky 1965; Linsky 1966; Troitsky, Burov & Aleshina 1967). The boundary condition is written in the form

$$(1-R_c)\,S_0 f(t)-(1-R_i)\;T^4(0,t)-\left[(K_0+K_r)\frac{\partial T}{\partial x}\right]_{x=0}=0. \tag{2}$$

Here S_0 is the density of the solar radiation flux, incident upon the body surface, R_i, R_c, the reflexion coefficients for the infrared and light waves, respectively (or the albedo in the case of a rough surface), σ, the Stefan–Boltzman constant. The first term in (2) corresponds to the energy absorbed at the surface, the second, to the radiation losses and the third to the heat flux coming from the body to the surface.

Yet, the introduction of K_r to the boundary condition such as (2) is not valid. Therefore, the set of equations (1) and (2) does not describe correctly the thermal régime of a surface layer of the material, being in vacuum and heated by the radiation; but they are widely used for the calculation of the thermal régime of the lunar surface.

The purpose of the present work is to derive the thermal conductivity equation together with the boundary conditions of the above problem in the presence of radiative energy transfer in the body.

Though in the literature there are derivations of the radiant thermal conductivity coefficient (Franck-Kamenetsky 1959; Filippov 1955), nevertheless, the problem is not analysed fully, taking into account the molecular thermal conductivity and the boundary conditions of the above mentioned type; the latter leads to the incorrect description of the boundary conditions (2) by analogy with a case of the molecular heat transfer only.

Derivation of equations

Let us consider both the plane-layer medium, all properties of which depend only on the coordinate x and the cylinder in it with the height dx and the unit bases. We write the equation of the energy balance in this element of the volume, considering the heat transfer through radiation and the molecular thermal conductivity.

As is known, the energy flux due to molecular thermal conductivity is equal to

$$F(x) = -K_0 \, \partial T/\partial x.$$

The radiant energy flux may be designated as $H(x)$. Apparently, the energy balance equation in the volume is expressed as

$$\frac{\partial}{\partial x}[F(x)+H(x)]+\rho c\frac{\partial T}{\partial t} = 0. \tag{3}$$

Then, the boundary conditions of the problem have the form

$$(1-R_c)\, S_0 f(t) - K_0 \left(\frac{\partial T}{\partial x}\right)_{x=0} + H(0) = 0. \tag{4}$$

The flux $H(x)$ is a function of the temperature and the properties of a medium. Then, the flux is equal to

$$H = 2\pi \int_0^\infty \mathrm{d}\nu \int_0^{\frac{1}{2}\pi} (I_{\nu+} - I_{\nu-}) \cos\vartheta \sin\vartheta \, \mathrm{d}\vartheta; \tag{5}$$

$I_{\nu+}$ is the spectral density of a medium radiation flux at a frequency ν, propagating deep into a body in the positive direction of an axis x, and $I_{\nu-}$, the same for the radiation in an opposite direction, ϑ, the angle between the direction of the radiation and the axis x.

As it is known, I_ν is the solution of the transfer equation. In this case, when a dense medium is considered, the principle of local thermodynamic equilibrium is assumed to be true under the condition that scattering is absent (the kinetic equilibrium of the particles within the element of the material). Then, the solution of the transfer equation contains the known Planck function B_ν. The coordinate x is expressed by an optical thickness

$$\tau_\nu(x) = \int_0^x \kappa_\nu(x) \sec\vartheta \, \mathrm{d}x,$$

where $\kappa_\nu(x)$ is an attenuation coefficient of the electromagnetic waves of the frequency ν in the medium. The radiation intensity at a point τ of the direction ϑ is equal to (the sign ν on τ_ν is omitted for simplicity):

$$\left.\begin{aligned} I_{\nu+}(\tau,\vartheta) &= n^2 \int_0^\tau B_\nu(\tau')\, \mathrm{e}^{-(\tau-\tau')} \mathrm{d}\tau', \\ I_{\nu-}(\tau,\vartheta) &= n^2 \int_\tau^\infty B_\nu(\tau')\, \mathrm{e}^{-(\tau'-\tau)}\, \mathrm{d}\tau'. \end{aligned}\right\} \tag{6}$$

Here, n is the refraction coefficient of the medium and $B_\nu = 2h\nu^3/c^2(e^{h\nu/kT}-1)$. An expression for H is an integral in a general case. Herein (3) becomes the integro-differential equation, the solution of which is difficult. It is possible to simplify significantly the expression on, assuming that the radiation intensity I at a point, x, is determined by the small volume around this point. This volume has a radius of the order of magnitude $l_\nu = 1/\kappa_\nu$. For the frequency, corresponding to the maximum B_ν, this magnitude has to be small relative to all the characteristic lengths of the problem. (This requirement is equivalent to that for full thermodynamic equilibrium in a medium, i.e. equilibrium not only within the material, but the material with radiation in a small element of the volume.) On this assumption $B(\tau')$ may be represented as a series in the vicinity of the point, x, in which only two terms of the expansion are essential (Filippov 1955):

$$B(\tau') = B(\tau) + (\tau'-\tau)\left(\frac{\partial B}{\partial \tau'}\right)_{\tau'=\tau}.$$

By substituting it into (6) we have

$$I_{\nu-}/n^2 = B(\tau) + \left(\frac{\partial B}{\partial \tau'}\right)_{\tau'=\tau} \quad \text{and} \quad I_{\nu+}/n^2 = B(\tau)\,(1-e^{-\tau}) - \left(\frac{\partial B}{\partial \tau'}\right)_{\tau'=\tau} + \left(\frac{\partial B}{\partial \tau'}\right)_{\tau'=\tau} e^{-\tau}(1+\tau).$$

The depth is assumed to be selected so far from the body surface, that $\tau(x) \gg 1$, then in the above expression the third term is much less then the second one and it may be neglected. Substituting both of the last expressions into (5) and bearing in mind that

$$\frac{\partial B}{\partial \tau} = l_\nu \cos\vartheta \frac{\partial B}{\partial x} = \frac{\partial B}{\partial T}\frac{\partial T}{\partial x} l_\nu \cos\vartheta,$$

one obtains the known expression for the radiation flux (see, for example, Filippov 1955) for $x > l_\nu$

$$H(x) = -\left[\frac{4\pi}{3}\int_0^\infty l_\nu \frac{\partial B}{\partial T} d\nu\right]\frac{\partial T}{\partial x} = -K_\tau \frac{\partial T}{\partial x}. \tag{7}$$

Now, we find $H(0)$. In this case, let $I_{\nu+} \equiv 0$ in (5). Therefore, instead of (6) the radiation coming from the body is written as

$$I_{\nu-}(0,\vartheta) = [1-R_\nu(\vartheta)]\int_0^\infty B_\nu(\tau')\,e^{-\tau'}d\tau', \tag{8}$$

where $R_\nu(\vartheta)$ is the reflexion coefficient of the surface for the radiation, falling inside the body. Assuming that $B_\nu(\tau') = B_\nu(0) + \tau'(\partial B/\vartheta\tau')_{\tau'=0}$ in (8) one gets

$$I_{\nu-}(0,\vartheta) = (1-R_\nu)\,[B_\nu(0) + (\partial B/\partial\tau')_{\tau'=0}].$$

Taking into account that $$\frac{\partial B}{\partial \tau} = l_\nu \cos\vartheta \frac{\partial B}{\partial T}\frac{\partial T}{\partial x},$$

we obtain according to (5):

$$-H(0) = 2\pi\int_0^\infty B_\nu(0)\,d\nu \int_0^{\frac{1}{2}\pi}(1-R_\nu)\cos\vartheta_1 \sin\vartheta_1\,d\vartheta_1$$
$$+2\pi\frac{\partial T}{\partial x}\int_0^\infty \frac{\partial B}{\partial T} l_\nu\,d\nu \int_0^{\frac{1}{2}\pi}(1-R_\nu)\cos\vartheta\cos\vartheta_1\sin\vartheta_1\,d\vartheta_1. \tag{9}$$

Here, ϑ_1 is the outer normal angle of the direction of radiation passed through the interface.

The angles ϑ_1 and ϑ are associated by the refraction law and

$$\cos\vartheta = \sqrt{\{1-n^{-2}\sin^2\vartheta_1\}}.$$

Let $$\int_0^{\frac{1}{2}\pi}(1-R_\nu)\cos\vartheta_1\sin\vartheta_1\,\mathrm{d}\vartheta_1 = (1-\bar{R}_\nu)\int_0^{\frac{1}{2}\pi}\cos\vartheta_1\sin\vartheta_1\,\mathrm{d}\vartheta_1 = \tfrac{1}{2}(1-\bar{R}_\nu)$$

and

$$\int_0^{\frac{1}{2}\pi}(1-R_\nu)\cos\vartheta\cos\vartheta_1\sin\vartheta_1\,\mathrm{d}\vartheta_1 = (1-\tilde{\tilde{R}}_\nu)\int_0^{\frac{1}{2}\pi}\cos\vartheta\cos\vartheta_1\sin\vartheta_1\,\mathrm{d}\vartheta = \tfrac{1}{3}(1-\tilde{\tilde{R}}_\nu)f(n),$$

where $f(n) = n^2[1-(1-n^{-2})^{\frac{3}{2}}]$. Then

$$H(0) = -\pi\int_0^\infty(1-\bar{R}_\nu)\,B_\nu(0)\,\mathrm{d}\nu - \left[\frac{2\pi}{3}\int_0^\infty(1-\tilde{\tilde{R}}_\nu)f(n)\,l_\nu\frac{\partial B}{\partial T}d\nu\right]\left(\frac{\partial T}{\partial x}\right)_{x=0}. \tag{10}$$

Here $\bar{R}_\nu$ and $\tilde{\tilde{R}}_\nu$ are the mean values of $R_\nu(\vartheta)$ for all the angles of ϑ with the weight functions of $\sin 2\vartheta_1$ and $\cos\vartheta\sin 2\vartheta_1$, respectively, having the maximum at the angles of 45° and inside the interval 35° to 45° depending on n^2. Therefore, the mean values are equal, approximately, to the values of R_ν at the corresponding angles; e.g. $\bar{R}_\nu \simeq R_\nu$ (45) and $\tilde{\tilde{R}}_\nu \simeq R_\nu$ (35). The value $f(n)$ varies in the interval $1 \leqslant f(n) \leqslant \frac{3}{2}$ at $1 \leqslant n \leqslant \infty$. Practically, even at $n^2 \geqslant 2$, $f(n) \simeq \frac{3}{2}$.

In the second term of (10) the value in the brackets has the sense of the radiant thermal conductivity coefficient for the radiation from the surface, expressed by K_{rs}. Let us simplify the expressions obtained. Usually, the values $\bar{R}_\nu$, $\tilde{\tilde{R}}_\nu$ and n vary little with frequency and it is possible to carry them outside the integral at the frequency values corresponding to the maximum of the radiation function. These values are expressed by $\bar{R}_i$, $\tilde{\tilde{R}}_i$ and n_i; then, if

$$\int_0^\infty B_\nu(0)\,\mathrm{d}\nu = \frac{\sigma}{\pi}T^4(0,t),$$

where σ is the Stefan–Boltzman constant, we obtain:

$$-H(0) = (1-\bar{R}_i)\,\sigma T^4(0,t) + K_{rs}\left(\frac{\partial T}{\partial x}\right)_{x=0}. \tag{11}$$

After the simple transformation one gets for K_r and K_{rs}

$$K_r = \tfrac{16}{3}n_i^2\,\bar{l}\sigma T^3 \tag{12}$$

$$K_{rs} = \tfrac{8}{3}(1-\tilde{\tilde{R}}_i)f(n)\,\bar{l}\sigma T^3 \tag{13}$$

where $$\bar{l} = \int_0^\infty l_\nu\frac{\partial B}{\partial T}\mathrm{d}\nu \Big/ \int_0^\infty\frac{\partial B}{\partial T}\mathrm{d}\nu$$

is the length of a quantum path of the thermal radiation averaged over all the frequencies in a given point of the medium. So, (3) and (4) may be written as

$$\frac{\partial}{\partial x}\left[(K_0+K_r)\frac{\partial T}{\partial x}\right] = \rho c\frac{\partial T}{\partial t} \quad (\bar{l}<x\leqslant\infty) \tag{14}$$

$$(1-R_c)\,S_0 f(t) - (K_0+K_{rs})\left(\frac{\partial T}{\partial x}\right)_{x=0} - (1-\bar{R}_i)\,\sigma T^4(0,t) = 0. \tag{15}$$

We see that the differential equation (14) is true for points of a body, further than $2\bar{l}$ to $3\bar{l}$ from the surface. The boundary condition (15) shows that the full radiation energy loss from the real surface, the material being penetrable for the waves of the self-radiation, may be represented as the sum of two losses: the radiation from the surface according to Stefan–Boltzmann law and a loss proportional to the temperature gradient on the surface. An expression (11) may be interpreted as

$$-H(0) \simeq (1-\bar{R}_i)\,\sigma\left[T(0,t)+\tfrac{2}{3}\bar{l}f(n)\left(\frac{\partial T}{\partial x}\right)_{x=0}\right]^4.$$

From this equation one may see that if a temperature gradient exists at the surface and the medium is penetrable for the self-radiation, the full radiation energy may be determined by the Stefan–Boltzmann law, but it is necessary to take the temperature value for some depth being equal to $\frac{2}{3}\bar{l}f(n)$; at $n = 1$ this depth is equal to $\frac{2}{3}\bar{l}$, and at $n^2 \geqslant 2$ it equals $\bar{l}$.

From equations (14) and (15) we find that the radiant thermal conductivity coefficient in the boundary conditions is $2n_i^2/(1-\bar{R}_i)f(n)$ times less than in a differential equation. Equations (1) and (2) are not correct for the solution of the thermal problem in the presence of radiative energy transfer due to the penetration of the material. Equations (14) and (15) are the correct system but the boundary of a medium tends to be out of the validity interval according to the argument x. Strictly speaking, both equations, however, do not form the system. They are related to different regions of the argument. In order to solve the problem, it is necessary to believe that (14) is valid over all the interval $0 \leqslant x \leqslant \infty$. This is an approximation which is more precise as $\bar{l}$ decreases in comparison with the characteristic lengths of the problems considered.

There arises the interesting situation when the radiant transfer takes place only through the pores, and the material itself remains impenetrable. If the pores do not come to the surface, $K_{\tau s} = 0$ and $K_\tau = l_\rho \sigma T^3$, where l_ρ is an effective size of the pores along the flux. The case of the pores, coming to the surface, is a very complicated one and requires a special consideration.

The calculation of the thermal Lunar régime, when taking into account the above-mentioned corrections, may lead (at the essential part of radiation thermal conductivity) to the change of the calculated night temperature and, therefore, to the change of the measured value $\gamma = (K\rho c)^{-\frac{1}{2}}$.

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